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A MONTE CARLO COMPARISON OF
NONPARAMETRIC RELIABILITY ESTIMATORS

by

Jia-Jinn Yueh

A report submitted in partial fulfillment
of the requirements for the degree

of

MASTER OF SCIENCE

in

Applied Statistics

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Approved:

UTAH STATE UNIVERSITY
Logan, Utah

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INTRODUCTION

It is very difficult to construct a reliability model for a complex system. However, the reliability model for a series configuration is relatively simple. In the simplest case in which the components are mutually independent, the system reliability can be represented as follows:

$$R_s(x) = \prod_{i=1}^n R_i(x),$$

where R_i is the reliability for the i^{th} component. It is also known that for moderate levels of system reliability for large systems, the component reliability must be high.

Extreme Value Theory indicates that under very general conditions, the initial form of the distribution function is the dominating contribution to the system reliability. In addition, under proper conditions (i.e., the component reliability distribution is such that its related extreme value type is of the limits type, [Gumbel, 1958]), the initial forms of these distributions are Weibull, i.e.,

$$1 - F(x) = \exp(-(x/\theta)^\gamma).$$

This distribution was proposed by W. Weibull in 1939 without mathematical foundation. Obviously, the Weibull distribution reduces to exponential distribution when the parameter γ equals one. Three typical Weibull distribution curves are drawn in Figure 1.

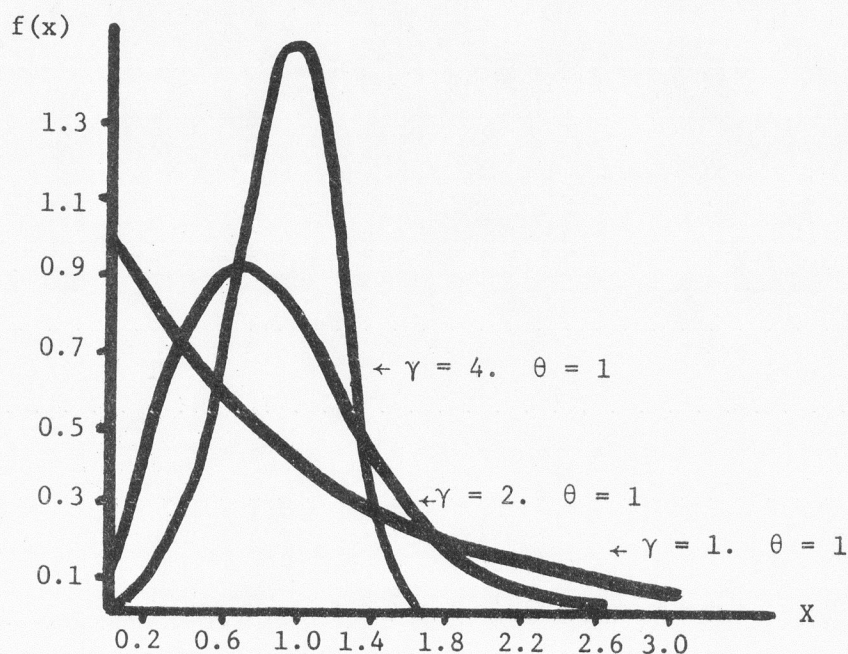


Figure 1. The density of the Weibull distribution for different values of the parameter γ and fixed θ .

These considerations suggest that the Weibull estimation techniques (see, for example, Bain and Antle, 1967) with suitable modifications may be useful nonparametric tools for estimating component reliabilities. Since the extreme value theory indicates that the initial shapes of the component distributions are Weibull, it seems appropriate to modify the standard Weibull estimation method such that the early time to failure data values are weighted heavier than later values. That is, the early time to failure data should have greater influence in determining the Weibull parameters than the later values.

It is expected that the shape of the component reliability functions is a Weibull distribution for early times but may depart significantly

for later times. Thus some sort of test of fit is needed to determine when the component distributions depart from the Weibull form. Previous studies of this problem have shown that the principle of invariance greatly facilitates the determination of the distribution of the test statistic. The general form of an invariant test statistic for this problem has been given by Canfield (1973). The Monte Carlo method is used to derive the distribution of the test statistic.

The main purpose of this paper is to compare the characteristics of reliability estimators which are obtained using three different levels of weighting factors on the estimation of Weibull parameters, (i.e., γ and θ), and three different levels of weighting factors on calculating the test statistic. Then, the bias and variance between test and true reliabilities are tabulated by using the Monte Carlo method. In this fashion, comparisons of the various weighting levels can be made. The obtained result is compared with Canfield's work to determine the appropriateness of each method.

THE TEST STATISTIC FOR TESTING THE DEPARTURE FROM A WEIBULL DISTRIBUTION

It is known that if a component reliability distribution is such that its related extreme value type is of the limites type, (Gumbel, 1958), its initial form will be close to the Weibull distribution. Thus the Weibull estimation techniques should give good estimates of the distribution function of the data are restricted to the time period in which the true distribution function is close to the Weibull form. In this section a statistic is developed for the purpose of testing for departure from a Weibull form.

The Estimation of Weibull Parameters

Estimates of the Weibull parameters γ and θ can be obtained by a method similar to that given by Bain and Antle (1967), i.e., by minimizing

$$\psi^* = \sum_{i=1}^k \omega_i [\ln(X_{(i)}/\theta)^\gamma - \ln(-\ln(1-(i-.5)/n))]^2$$

where n is the sample size ($n = 100$ in this paper), k is the number of data points used in calculating the parameters, and ω_i is the weighting factor. The value k is referred to as the truncation value.

Three different weighting factors are considered, i.e.,

$$\omega_i = \exp(m(100-i)/100),$$

for $m = 3, 4.5, 6$. Note that $\omega_i > \omega_j$ for $i > j$. The results obtained here are compared with each other and with previous methods studied by Canfield (1973) in order to acquire a more accurate estimation procedure.

The estimators of the parameters of the Weibull form

$$F(x) = 1 - \exp(- (x/\theta)^\gamma)$$

are found as follows:

$$\hat{\gamma} = \frac{\sum_{i=1}^k \omega_i \ln(X_{(i)}) \ln(q_i) - \frac{\sum_{i=1}^k \omega_i \ln(X_{(i)}) \sum_{i=1}^k \omega_i \ln(q_i)}{T}}{\sum_{i=1}^k \omega_i (\ln(X_{(i)}))^2 - \left(\sum_{i=1}^k \omega_i \ln(q_i) \right)^2 / T}$$

and

$$\hat{\theta} = \frac{\prod_{i=1}^k (X_{(i)})^{\omega_i/T}}{\prod_{i=1}^k q_i^{\omega_i/(T\hat{\gamma})}}$$

where

$$q_i = - \ln(1-(i-.5)/n)$$

$$T = \sum_{i=1}^k \omega_i.$$

These estimators of the Weibull parameters are calculated in this paper for truncation values (i.e., k) from 15 to 60 in steps of 5 for convenience and economy. The above estimates $\hat{\gamma}$ and $\hat{\theta}$ are recalculated at each truncation value k .

The Test Statistic $h^*(x,k)$

The principle of invariance is a useful tool in hypothesis testing (Lehmann, 1959). If a test statistic which is invariant to the parameters of the hypothesized population can be found, then the distribution of the test statistic is independent of those parameters. This greatly simplifies the distribution of the test statistic. The following theorem (Canfield, 1973, p. 127) is useful in establishing the test statistic for the tests required in this paper. Invariance, in this case provides a test statistic which is independent of the population parameters.

Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be the order statistics from a sample size n of a Weibull distributed random variable by γ and θ respectively. Let $\hat{\gamma}$ and $\hat{\theta}$ be the estimates of γ and θ obtained by minimization of equation

$$\psi^* = \sum_{i=1}^k \omega_i [\ln(X_{(i)}/\theta)^\gamma - \ln(-\ln(1-(i-.5)/n))]^2$$

for some $k < n$. For convenience define $\eta(X_{(i)}) = (X_{(i)}/\hat{\theta})^{\hat{\gamma}}$.

The distribution of $h(\eta(X_{(1)}), \eta(X_{(2)}), \dots, \eta(X_{(k)}))$ is independent of γ and θ where h is an arbitrary function.

The statistic used in this work to test for departure from a Weibull is of the form:

$$h^*(x,k) = \sum_{i=1}^k \omega_i (f_i - f'_i)/k$$

where

$$f_i = 1 - F_e(X_{(i)}) = 1 - (i-.5)/n \dots \dots \dots (1)$$

(F_e is the empirical distribution function (Gibbons, 1971))

$$\begin{aligned}
 f'_i &= 1 - F(X_{(i)}) \\
 &= \exp \left(- (X_{(i)} / \hat{\theta})^{\hat{\gamma}} \right),
 \end{aligned}$$

k is the truncation value at which $\hat{\gamma}$ and $\hat{\theta}$ were calculated, and

$$\omega'_i = \exp (m(i)/100)$$

will be used as a weighting factor at three different levels, say, $m = 3, 4.5, 6$. Note that $h^*(x, k)$ satisfies the criterion in the previous theorem, and therefore is independent of the parameters γ and θ . The function used here has a simpler form than that used by Canfield (1973), and therefore may give better results.

If the null hypothesis for testing the departure is chosen to be $H: X \in G$, where G is a family of Weibull distributed random variables, the test will be of the form

$$\phi(h^*(x, k)) = \begin{cases} 1 & \text{if } h^*(x, k) \geq K \text{ (reject)} \\ 0 & \text{if } h^*(x, k) < K \text{ (accept)} \end{cases}$$

where K is a specified constant.

This specified K is determined from the distribution of $h^*(x, k)$ and is a function of the desired α level of the test. The distribution function for $h^*(x, k)$ is difficult to be determined mathematically. However, Monte Carlo simulation is relatively easy. The later method is used in this paper.

Distribution of Test Statistic

Monte Carlo method

The Monte Carlo method is based on random sampling within the boundaries set for the variables of the problem. Behavior of physical quantities, of man-machine systems, of technological ensembles, etc., can be simulated through Monte Carlo techniques. In this paper, the distribution of $h^*(x,k)$ is needed.

The Monte Carlo method gives empirical distribution functions taken from observational data, which are as easy to use as exact mathematical functions. However, the accuracy of these functions as approximations to the true distributions may be questioned. It is known that the empirical distribution function (1) converges to the true distribution function as the sample size increases (Gibbons, 1971). Thus for large sample size the approximation should be good. In computer simulation the sample size is determined primarily by the computer budget.

Confidence bands for the true distribution function $F(x)$ for all x can be found by the Kolmogorov-Smirnov one-sample statistic D_n , which is defined as $D_n = \sup_x |F_e(x) - F(x)|$ where F_e is the empirical distribution function of a random sample $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ from the distribution function $F(x)$. Another number $D_{n,\alpha}$ can be found such that

$$P(D_n > D_{n,\alpha}) = \alpha.$$

This is equivalent to the statement

$$P(\sup_x |F_e(x) - F(x)| < D_{n,\alpha}) = 1 - \alpha$$

or

$$P(F_e(x) - D_{n,\alpha} < F(x) < F_e(x) + D_{n,\alpha}) = 1 - \alpha.$$

Since $0 \leq F(x) \leq 1$ for all x , and the above inequality in the probability statement admits some numbers outside this range, an improved statement is as follows. Let

$$L_n(x) = \max(F_e(x) - D_{n,\alpha}, 0)$$

and

$$U_n(x) = \min(F_e(x) + D_{n,\alpha}, 1).$$

Then $L_n(x)$ and $U_n(x)$ are the confidence limits for $F(x)$, and the region between $L_n(x)$ and $U_n(x)$ is called a confidence band for $F(x)$ (Gibbons, 1971).

For $\alpha = 0.05$, the confidence band of true distribution has the length $2(1.36)/\sqrt{N}$, (Ostle, 1963), where N is the number of times in simulation. If N equals 2,000, the confidence band has the length of 0.0608. In this manner, the accuracy of the sample distribution function of the test statistic as an approximation to the population distribution function can be judged. In this paper, it was possible to use only 500 simulated observations because of budgetary limitations. The width of the confidence band for $N = 500$ is 0.1212. Thus the results may not be taken as extremely accurate but they will give a good indication of the difference between estimation methods.

Distribution of $h^*(x,k)$

Since the distribution of the test statistic chosen in the Test Statistic section is independent of the population parameters under the Weibull assumption, the distribution of the test statistic is the same for all Weibull parameters. Therefore, the simple exponential form of the Weibull distribution ($F_1(x) = 1 - \exp(-x)$) is used to generate the required data in those Monte Carlo derivations.

A set of order statistic $\{X_{(i)}, i=1, 2, \dots, 100\}$ is generated from $F_1(x)$ by using the method given by Lurie and Hartley (1972). The basic distributional facts that are employed in this method are as follows:

The c.d.f. of $b_{(1)}$ is given by

$$P\{b_{(1)} \leq B_1\} = 1 - (1 - B_1)^n \dots\dots\dots (2)$$

where $b_{(i)}$ are computed from the ordered statistic in a sample of size n drawn from a population with c.d.f. $F_1(x)$.

It follows that $b_{(i)}$ can be generated directly with the help of a uniform variable V_1 by computation

$$b_{(i)} = 1 - V_1^{1/n} \dots\dots\dots (3)$$

The conditional c.d.f. of $b_{(i+1)}$ given $b_{(i)}$ is given by

$$P\{b_{(i+1)} \leq B_{i+1} | b_{(i)}\} = 1 - ((1 - B_{i+1}) / (1 - b_{(i)}))^{n-1} \dots\dots (4)$$

It follows that $b_{(i+1)}$ can be generated from $b_{(i)}$ with the help of a uniform variable V_{i+1} through

$$b_{(i+1)} = 1 - (1 - b_{(i)})^{V_{i+1}^{1/(n-1)}} \dots\dots\dots (5)$$

Equations (3) and (5) from the basis of an n-step loop subroutine which will generate the $b_{(i)}$ sequentially.

In this problem, the random sample is $B(x) = 1 - \exp(-x)$, whose inverse function $B^{-1} = x = -\ln(1 - B)$, is used in the following.

Such data are used through the procedures developed in The Estimation of Weibull Parameters and The Test Statistic sections to find the values of the test statistic at each truncation value.

The entire calculation for each combination of ω_i and ω'_i ($i = 1, 2, 3$) is repeated 500 times. Keeping the biggest one hundred values of $h^*(x, k)$ at each truncation value k , the 90th, 85th, and 80th percentile, with respect to the 50th, 75th, and 100th biggest, of the $h^*(x, k)$ value will be located beneath the point obtained. Use of these probability curves will be discussed in the next section.

The computer program for the above procedure is listed in Appendix A, and the first seven sets of curves are drawn in order (Figures 2 to 8).

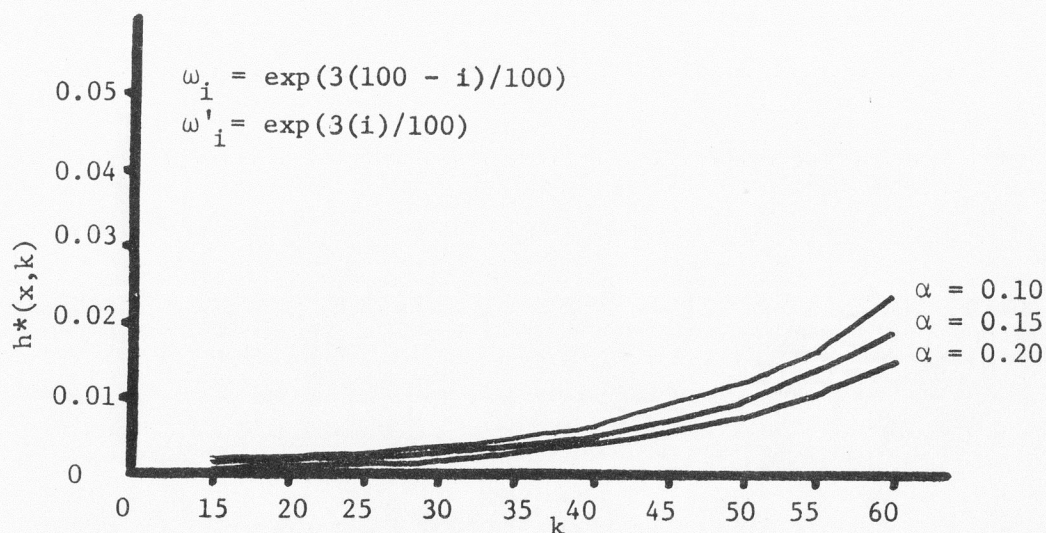


Figure 2. Probability curves of $h^*(x, k)$. (For the combination of $m = 3$ in ω_i and $m = 3$ in ω'_i .)

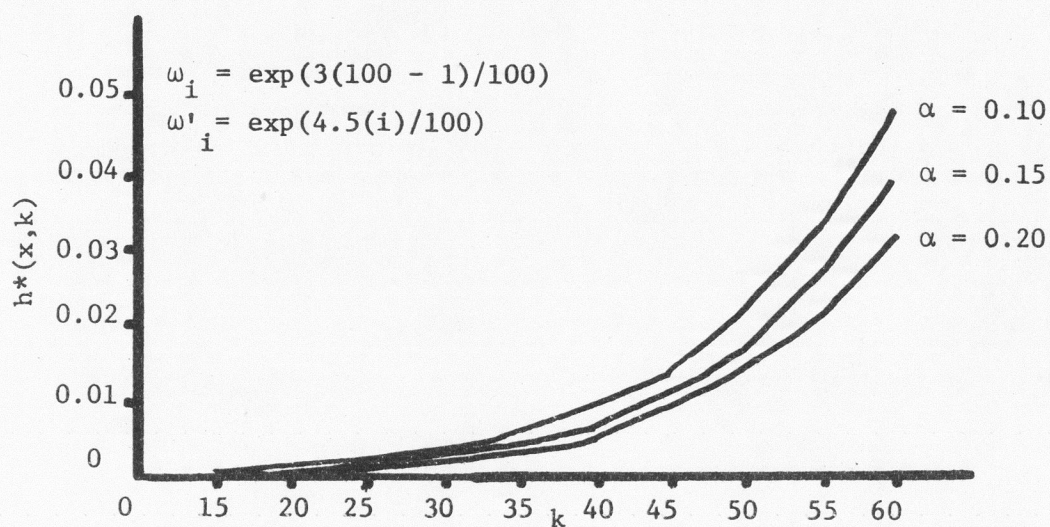


Figure 3. Probability curves of $h^*(x,k)$. (For the combination of $m = 3$ in ω_i and $m = 4.5$ in ω'_i .)

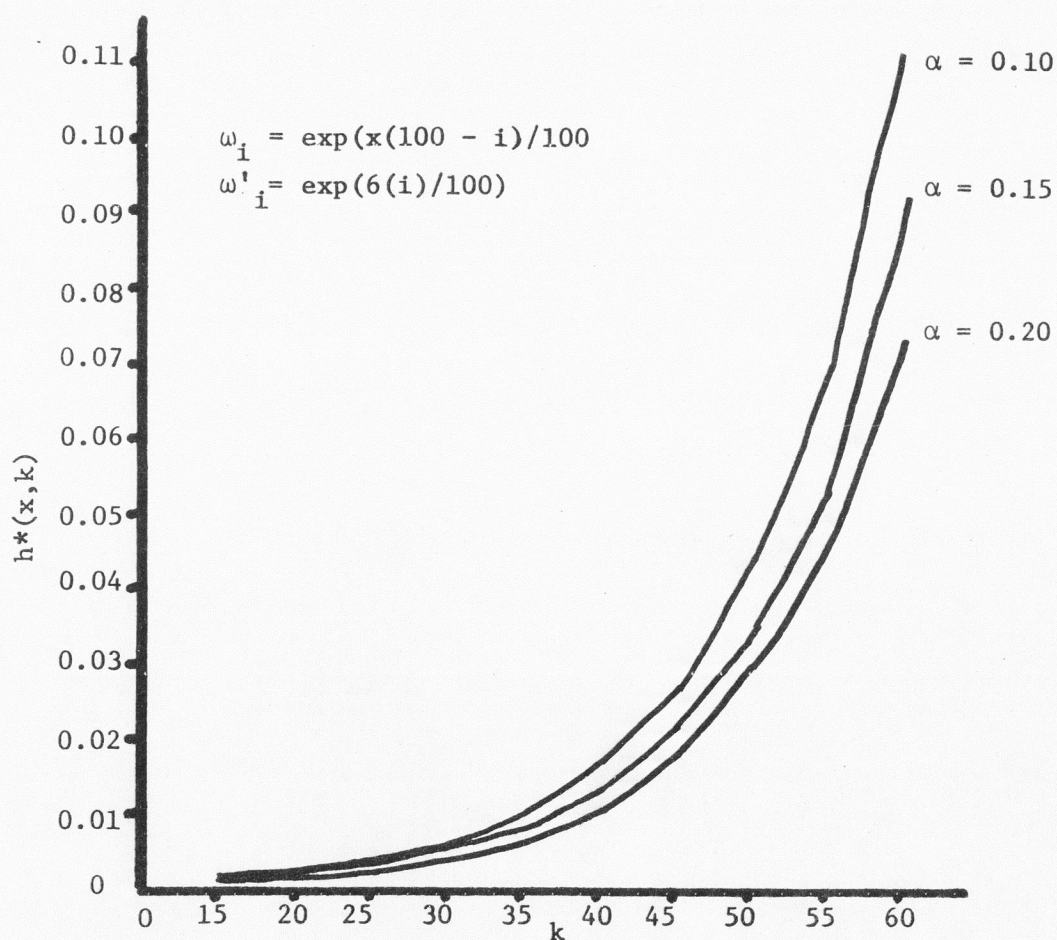


Figure 4. Probability curves of $h^*(x,k)$. (For the combination of $m = 3$ in ω_i and $m = 6$ in ω'_i .)

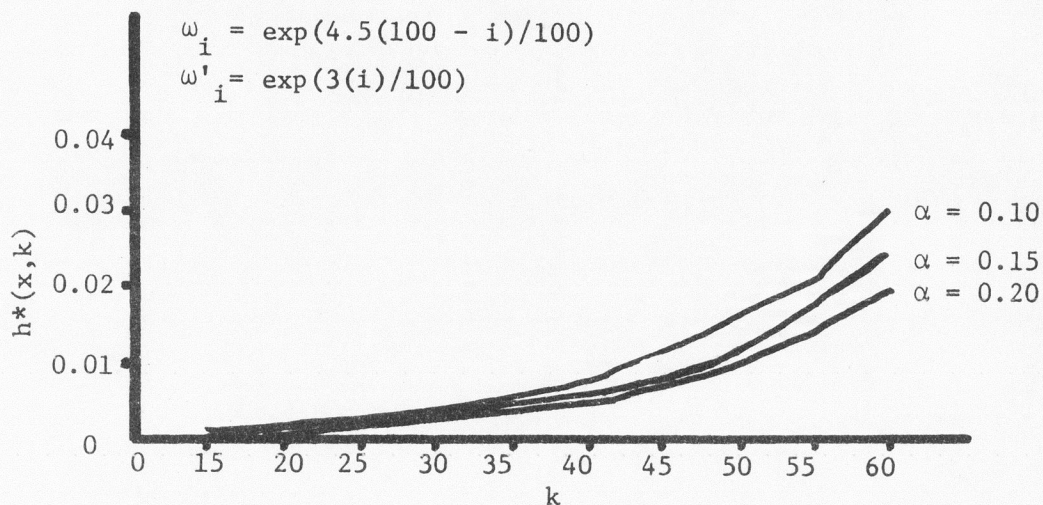


Figure 5. Probability curves of $h^*(x,k)$. (For the combination of $m = 4.5$ in ω_i and $m = 3$ in ω'_i .)

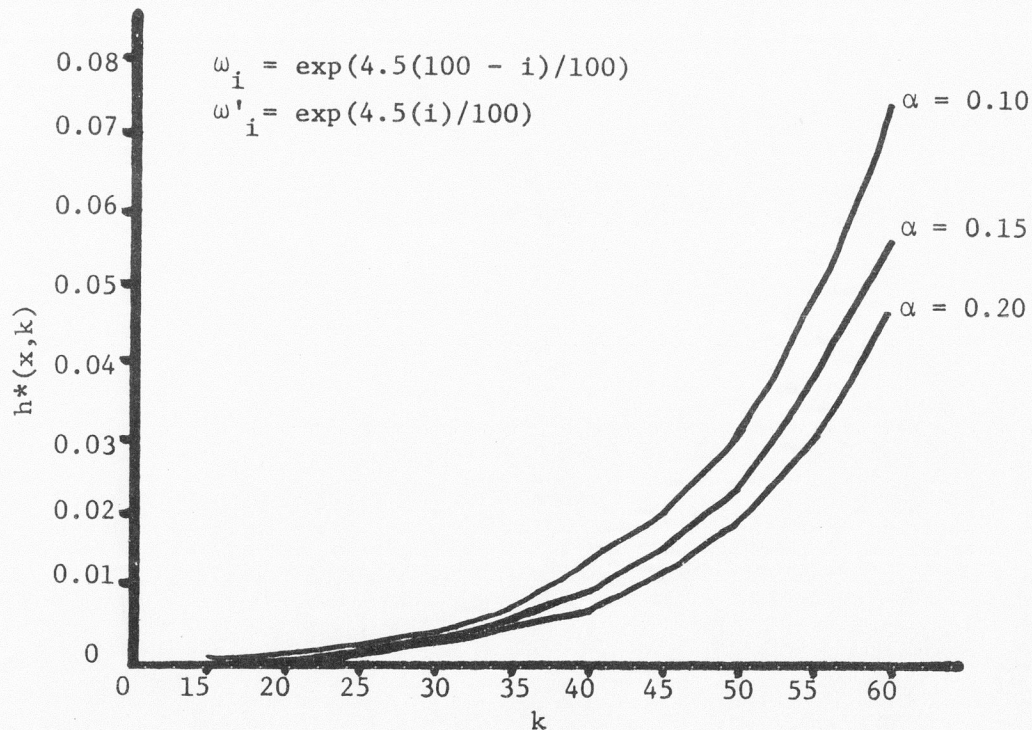


Figure 6. Probability curves of $h^*(x,k)$. (For the combination of $m = 4.5$ in ω_i and $m = 4.5$ in ω'_i .)

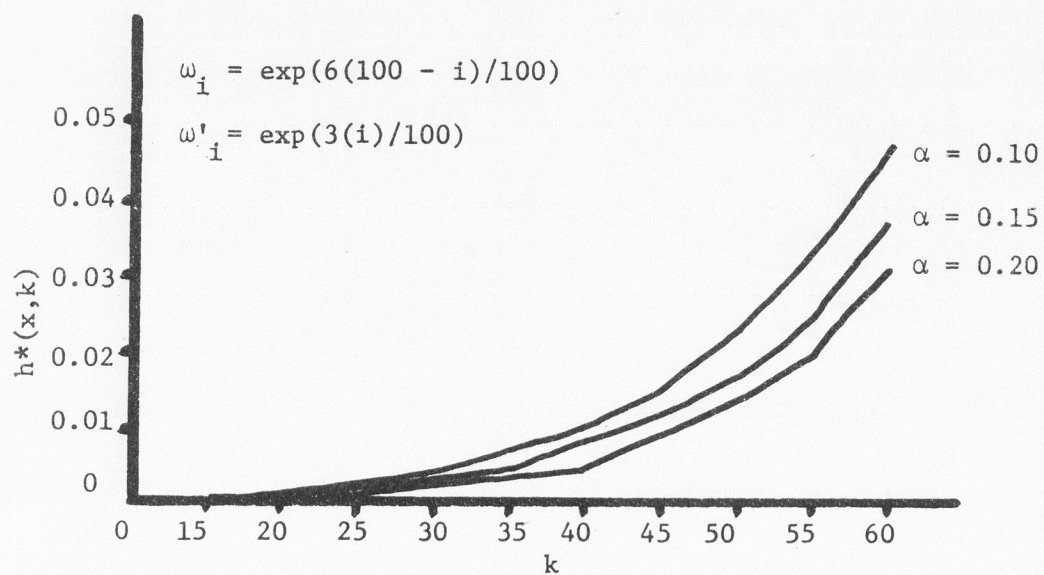


Figure 7. Probability curves of $h^*(x,k)$. (For the combination of $m = 6$ in ω_i and $m = 3$ in ω'_i .)

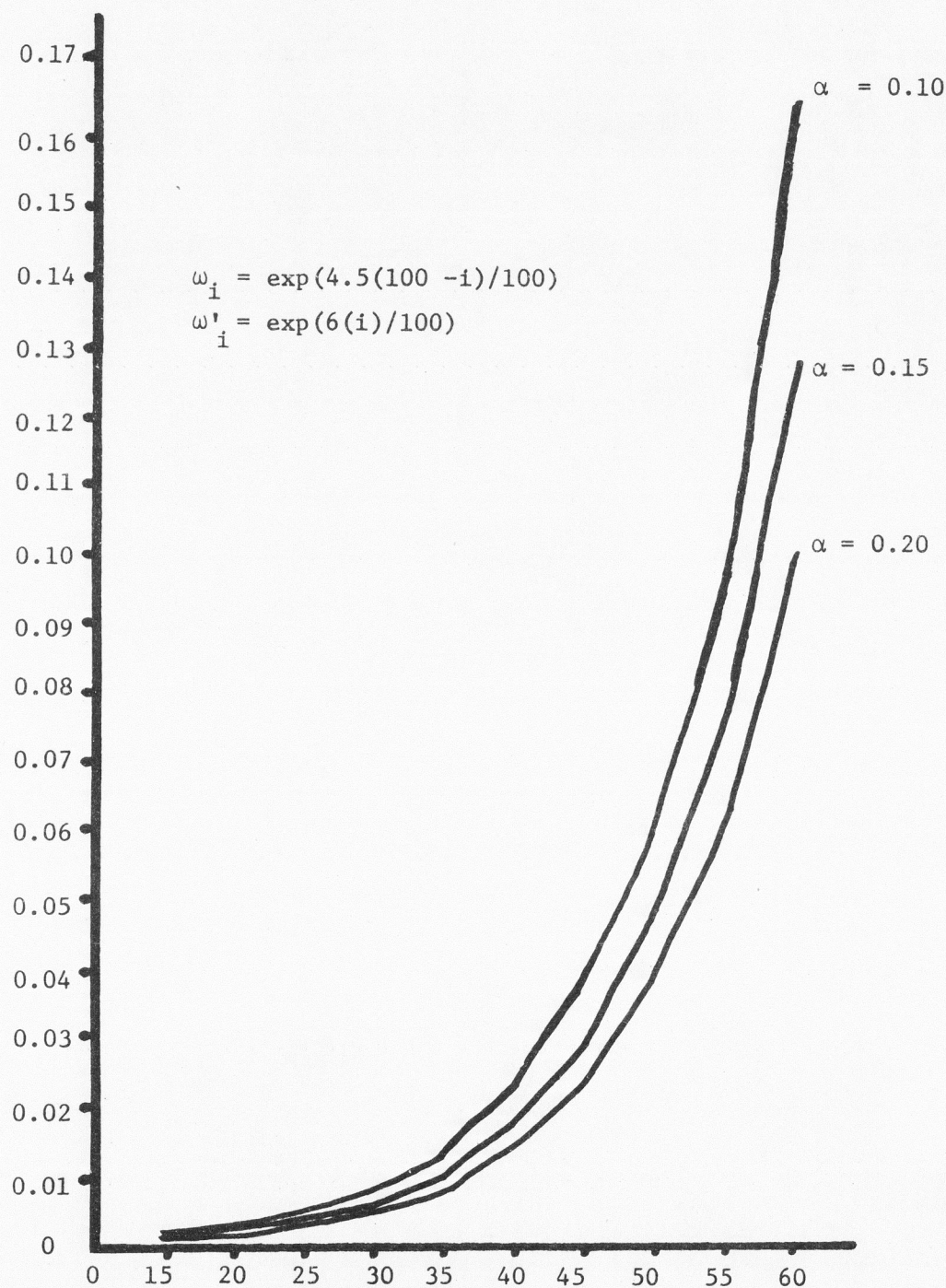


Figure 8. Probability curves of $h^*(x,k)$. (For the combination of $m = 4.5$ in ω_i and $m = 6$ in ω'_i .)

COMPARISON OF NONPARAMETRIC RELIABILITY ESTIMATORS

This section utilizes hazard functions to determine the break-point. And then choosing some points on both side of the break-point to be data points from which the bias and variance can be calculated. A distribution function $F_2(x) = 1 - e^{-.2x})(1 - e^{-.8x})$, not belonging to the Weibull family, is chosen to be a true component reliability density function for making the comparison. Thus, the data set (or time series) will be generated from $F_2(x)$ by the method employed in the section The Test Statistic for Testing the Departure from a Weibull Distribution. Let $x = \max(x_1, x_2)$, then

$$\begin{aligned} F_2(x) &= P\{\max(x_1, x_2) \leq x\} \\ &= F_a(x)F_b(x) \\ &= (1 - e^{-.2x})(1 - e^{-.8x}). \end{aligned}$$

So, the inverse functions associated with $F_2(x)$ are

$$B^{-1} = x_1 = (-5)(\ln(1 - B))$$

and

$$B^{-1} = x_2 = (-5/4)(\ln(1 - B))$$

$\max(x_1, x_2)$ is chosen as the data points hereafter.

Hazard Function and Break-point

The hazard function, $H(x)$, for random variable x is written as $(f(x)/(1 - F(x)))$ or $(-R'(x)/R(x))$ where $R(x)$ denotes the reliability function. It is obvious that $H(x)$ is a function of time, denoting the failure rate as time elapses.

For $F_2(x)$, the reliability

$$\begin{aligned} R_2(x) &= 1 - F_2(x) \\ &= 1 - (1 - e^{-.2x})(1 - e^{-.8x}) \\ &= e^{-.2x} + e^{-.8x} - e^{-x}, \end{aligned}$$

and

$$\begin{aligned} f_2(x) &= F'_2(x) \\ &= .2 e^{-.2x} + .8 e^{-.8x} - e^{-x}. \end{aligned}$$

By definition, the hazard function $H(x)$ is:

$$\begin{aligned} H(x) &= f_2(x)/(1 - F_2(x)) \\ &= (.2 e^{-.2x} + .8 e^{-.8x} - e^{-x})/(e^{-.2x} + e^{-.8x} - e^{-x}) \\ &= e^x(.2 e^{-.2x} + .8 e^{-.8x} - e^{-x})/e^x(e^{-.2x} + e^{-.8x} - e^{-x}) \\ &= (.2 e^{.8x} + .8 e^{.2x} - 1)/(e^{.8x} + e^{.2x} - 1). \end{aligned}$$

The shape of this hazard function is drawn as in Figure 9.

Some typical shapes of the hazard function for the Weibull distribution are shown in Figure 10.

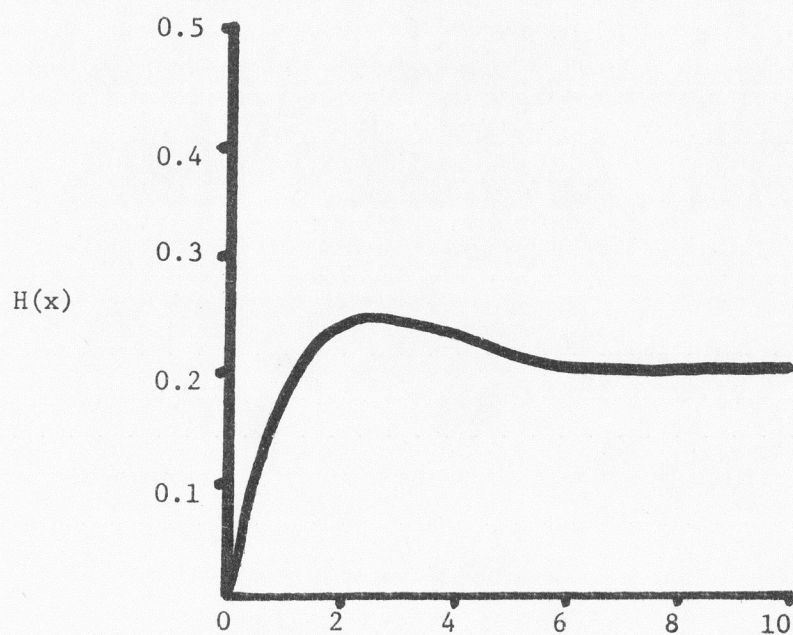


Figure 9. Hazard function of $F_2(x)$.



Figure 10. Hazard functions of Weibull distribution.

Comparing hazard function of $F_2(x)$ in Figure 9 with those in Figure 10, shows that after $x = 2$, the hazard function of $F_2(x)$ is quite different with those shapes of hazard function of the Weibull distribution. Thus, $F_2(x)$ departs significantly from $F(x)$ when $x > 2$. So the point $x = 2$ at which $F_2(x)$ begins to depart from the Weibull distribution function $F(x)$ is called a break-point. For this reason, three points, $x_1 = 0.27$, $x_2 = 1$, and $x_3 = 3$ are chosen for the comparison of the estimators.

Testing for the Hypothesis

The null hypothesis $H: X \in G$ where G is a family of Weibull distributed random variables is defined as before. For this test, the procedure as in The Test Statistic section is used with the new set of data generated from $F_2(x)$. The value of the test statistic is calculated at truncation values from 15 to 60 in steps of 5. This procedure will stop whenever a α level significance occurs, that is, the test statistic is greater than the $h^*(x,k)$ value at the same truncation value k defined in the α level probability curves. For the same set of data, there are nine different combinations of weighting factors to be tested. If the significance occurs at the i^{th} data point, the decision is made to accept the hypothesis of an approximate Weibull distribution for the population up to and including the $(i-5)^{\text{th}}$ order statistic and not Weibull beyond that value of time. So the first $(i-5)$ data points are used to calculate the Weibull parameters γ and θ . For example, let the test statistic at $k = 15, 20, 25, 30$ be 1.2, 1.8, 1.5, and 3.5, respectively. The probability curve is shown in Figure 11.

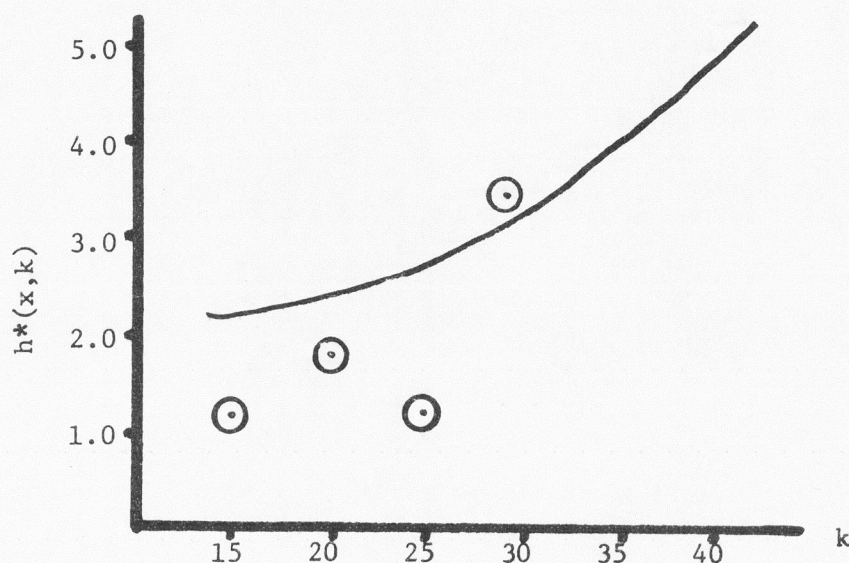


Figure 11. Probability curve for the example.

Those points indicated by circles are the values of the test statistic. At $k = 30$, it is significant because the test statistic goes above the probability curve. So $k = 25$ will be the truncation value for the data set, i.e., $X_{(i)}$, $i=1, 2, \dots, 25$, to estimate the Weibull parameters γ and θ . The estimation will be more accurate if the null hypothesis is tested in steps of unity instead of 5 as was done here.

Comparison of the Estimators

The true reliability $R_2(x)$ with density function $F_2(x)$ at three different time-points chosen in the Hazard Function and Break-point section are:

$$R(x_i) = 1 - F(x_i)$$

$$\begin{aligned}
 &= 1 - (1 - \exp(-(x_i/\hat{\theta})^{\hat{\gamma}})) \\
 &= \exp(-(x_i/\hat{\theta})^{\hat{\gamma}})
 \end{aligned}$$

where $i = 1, 2, 3$.

The bias of reliabilities at these three points is

$$a_i = \sum_1^M (R(x_i) - R_2(x_i))/M$$

where $i = 1, 2, 3$ and M is the number of times the test is simulated.

The variance of the test reliability will be

$$v_i = [\sum_1^M R^2(x_i) - (\sum_1^M R(x_i))^2/M]/(M - 1)$$

where $i = 1, 2, 3$ and M is the number of times the test is simulated.

The above procedure should be run 500 times to obtain the bias and variance. The computer program is listed in Appendix B. After running through 25 times the result for the first seven combinations are tabulated as follows (Table 1).

In Table 1, the combination of $\omega_i = \exp(6(100 - i)/100)$ and $\omega'_i = \exp(3i/100)$ seems to give the best result for this work. So the combination of $m = 6$ in ω_i and $m = 3$ in ω'_i is chosen to compare with the previous work (Canfield, 1973). Using the same procedure as above, $F_3(x) = (1 - e^{-.3x})(1 - e^{-.7x})$ is used to generate the data. The bias and variance at time 0.01, 0.5, and 1 are tabulated in Table 2.

In the previous work 1000 simulations were used to calculate the bias and variance. So the mean bias could be treated as a population mean bias. The previous results are tabulated in Table 3.

Table 1. Bias and variance of the test reliability

		$\omega'_i = \exp(mx_i/100)$						
		$m = 3$		$m = 4.5$		$m = 6$		
		Bias	Variance	Bias	Variance	Bias	Variance	
$\omega_i = \exp(mx(100-i)/100)$	$m = 3$	$x_1 = 0.27$	- 0.03936	0.00047	- 0.03935	0.00047	- 0.03934	0.00047
		$x_2 = 1$	- 0.10217	0.00439	- 0.10256	0.00439	- 0.10262	0.00437
		$x_3 = 3$	- 0.13531	0.04401	- 0.13692	0.04424	- 0.13733	0.04421
	$m = 4.5$	$x_1 = 0.27$	- 0.03910	0.00047	- 0.03910	0.00047	- 0.03912	0.00047
		$x_2 = 1$	- 0.10176	0.00469	- 0.10189	0.00469	- 0.10205	0.00473
		$x_3 = 3$	- 0.13296	0.04615	- 0.13449	0.04604	- 0.13687	0.04625
	$m = 6$	$x_1 = 0.27$	- 0.03878	0.00046				
		$x_2 = 1$	- 0.10158	0.00503				
		$x_3 = 3$	- 0.13161	0.04846				

Table 2. The bias and variance of the test reliability by using the data from $F_3(x)$. (Only 25 simulations.)

Time	Bias	Variance
0.01	- 0.00397	2.2×10^{-5}
0.5	- 0.10461	1.7×10^{-3}
1	- 0.18003	1.9×10^{-2}

Table 3. Bias and variance from the previous results by Canfield (1973)

Time	Bias	Variance
0.01	- 0.000082	4.5×10^{-6}
0.5	- 0.004	2.6×10^{-4}
1	- 0.019	8.3×10^{-4}

Because the resources for computer work was limited the following was used to determine the usefulness of the methods being considered.

The 95% confidence interval is defined as $\bar{a} \pm t_{24, .05/2}(\sqrt{v}/\sqrt{N})$ for the bias where N is the number of times the test is simulated. So the 95% confidence intervals for the bias in Table 2 at times 0.01, 0.5, and 1 are $(- 0.00199, - 0.00595)$, $(- 0.08723, - 0.12199)$, and

(- 0.12213, - 0.2393), respectively. All of these confidence intervals fail to cover and are greater than the previous results at times 0.01, 0.5, and 1 in Table 3, respectively. Thus it is very likely that this method does not give better results than the previous work, so it is not valuable to run further.

CONCLUSIONS

After comparing the characteristics of reliability estimators which were obtained using three different levels of weighting factors on the estimation of Weibull parameters, (i.e., γ and θ), and three different levels of weighting factors on calculating the test statistic, the result showed that the combination of the weighting factors $\omega_i = \exp(6(100 - i)/100)$ and $\omega'_i = \exp(3(i)/100)$ was the best of the seven results obtained.

By the analysis of the bias of the test reliabilities, it was found that the method presented did not give better results than the previous work (Canfield, 1973).

The following is a possible explanation for the apparent superiority of the method of Canfield (1973). Figures 12, 13 and 14 show plots of the distribution function, hazard function and log reliability, respectively of a non-Weibull distribution. It also shows the respective plots of a Weibull distribution which is initially close to the non-Weibull distribution function.

By the definition of distribution function, $0 \leq F(x) \leq 1$ and $0 \leq F_2(x) \leq 1$, the curves in Figure 12 will converge to 1 as time elapses, so that a maximum difference is reached for some time point. But those curves in Figures 13 and 14 will never converge as time elapses. Comparing those curves in Figure 12 and Figure 14 implies that the discrepancy between the distribution functions $F(x)$ and $F_2(x)$ is

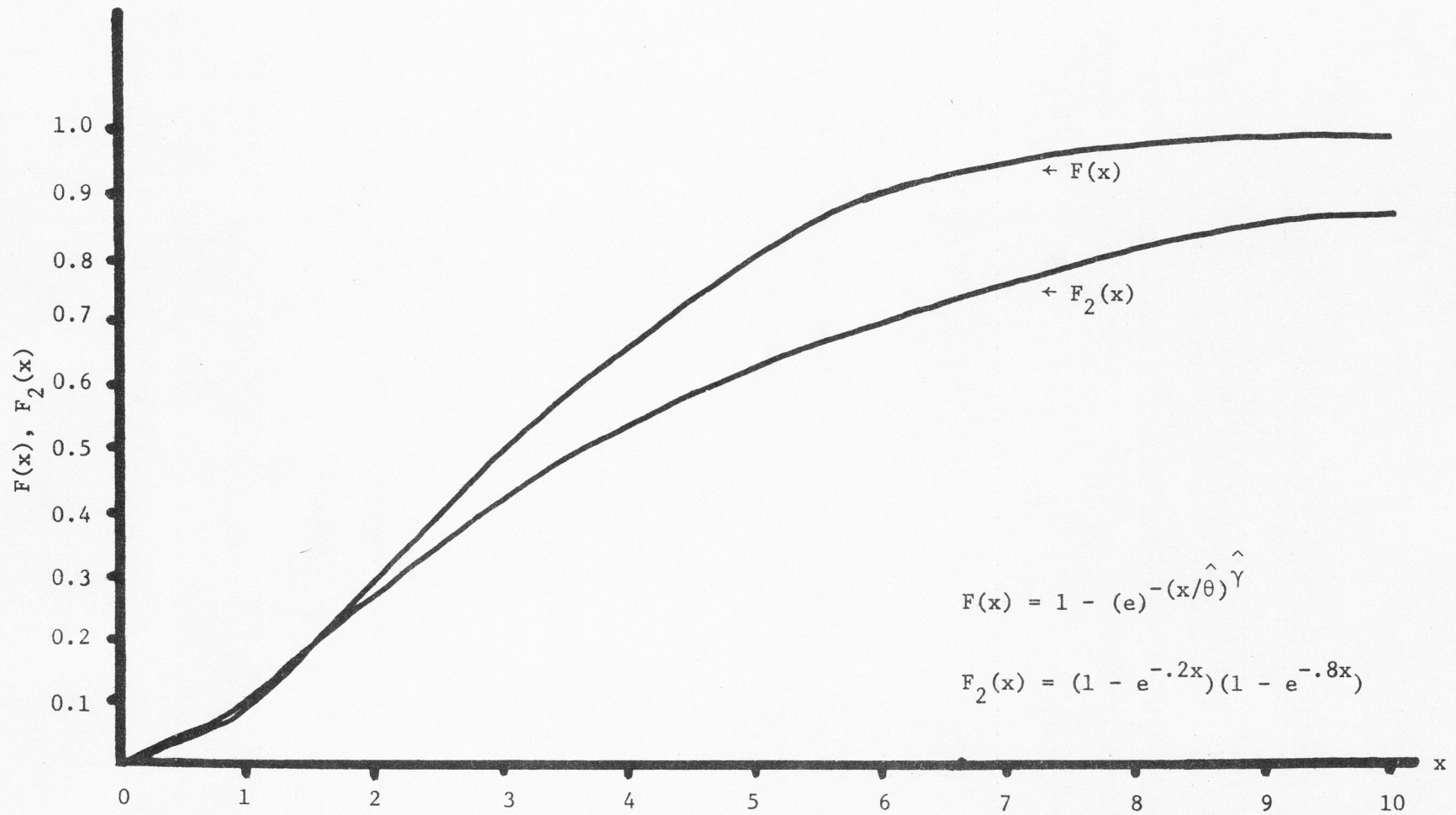


Figure 12. Distribution functions of $F(x)$ and $F_2(x)$.

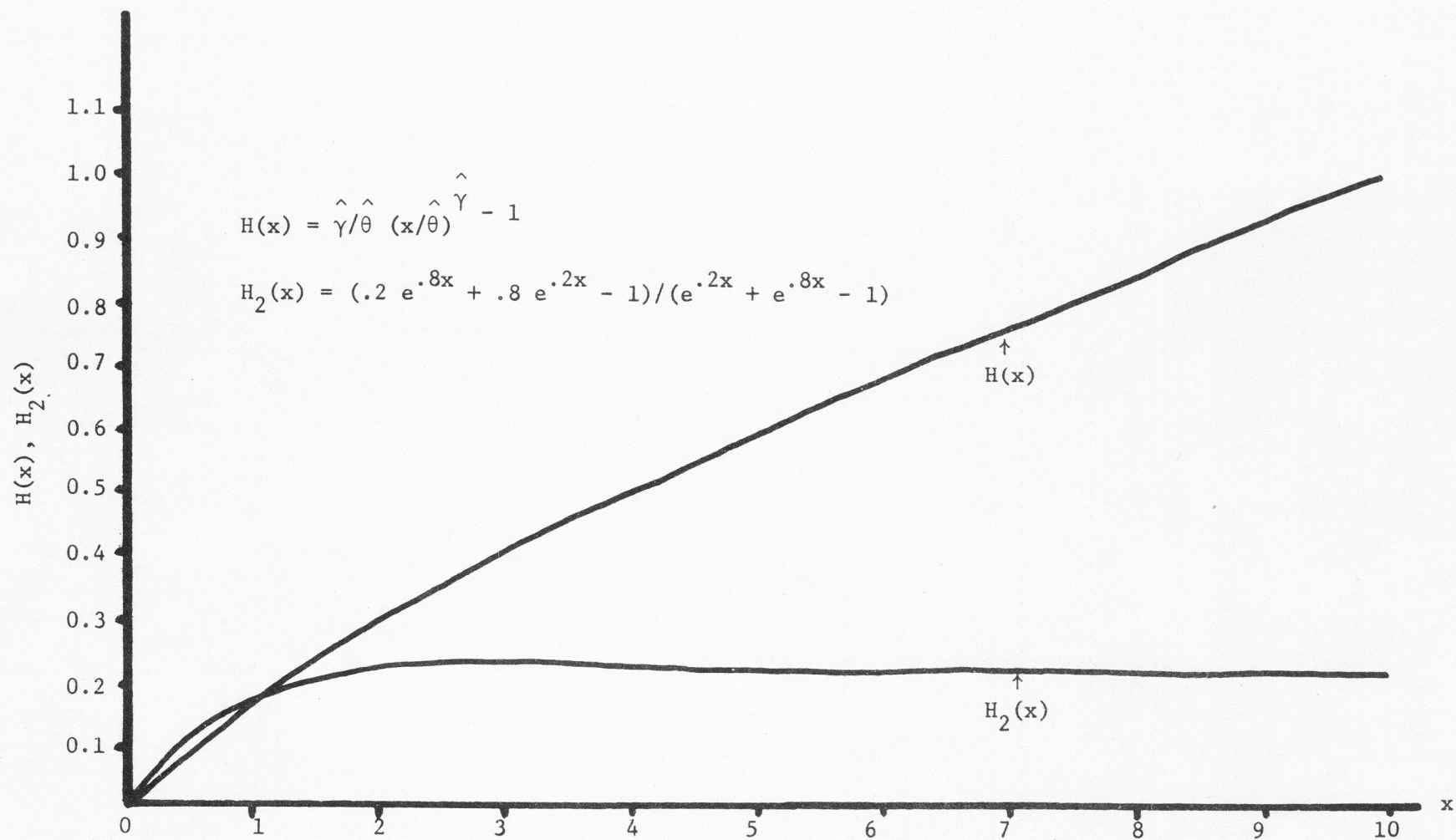


Figure 13. Distribution functions of $H(x)$ and $H_2(x)$.

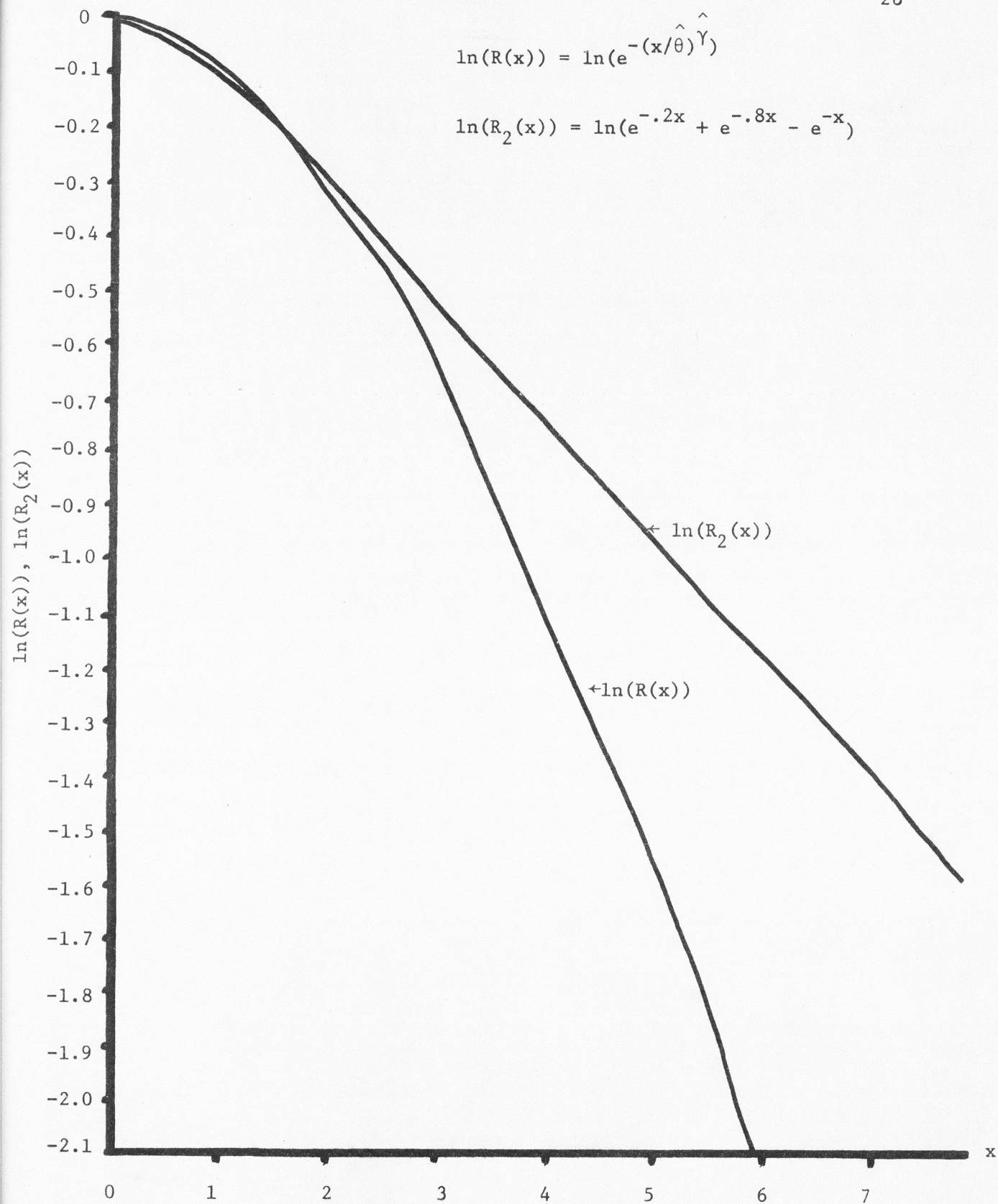


Figure 14. Distribution functions of $\ln(R(x))$ and $\ln(R_2(x))$.

smaller than that of $\ln(R(x))$ and $\ln(R_2(x))$ which were used in previous work. So it seems intuitively that a better result should be obtained from the method of Canfield (1973) which uses log reliability to measure departures from a Weibull. In Figure 13, the hazard functions $H(x)$ and $H_2(x)$, where $H(x)$ and $H_2(x)$ are the hazard functions of $R(x)$ and $R_2(x)$, respectively, depart earlier than those curves, $\ln(R(x))$ and $\ln(R_2(x))$, in Figure 14. Therefore, it is suggested that the test statistic $h^*(x,k)$ would be more appropriate if the hazard functions of the reliability and Weibull curves rather than the distribution or the log reliability curves themselves were used to make the comparison.

Unfortunately the hazard function is not of the proper form to insure invariance of the test statistic. However further study may produce a form which is invariant and which incorporates the intuitive advantages of the hazard function as a means of comparison.

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APPENDICES

Appendix A

MAIN PROGRAM

```
COMMON X(100),N,H(60,101),LARGE,T,U,V,Y,Z
```

```
DIMENSION A(3)
```

```
N=100
```

```
LARGE=5114369
```

```
A(1)=3.
```

```
A(2)=4.5
```

```
A(3)=6.
```

```
D) 200 I=1,3
```

```
DO 200 J=1,3
```

```
P1=A(I)
```

```
P2=A(J)
```

```
DO 100 JJ=1,500
```

```
J1=JJ
```

```
T=0.
```

```
U=0.
```

```
V=0.
```

```
Y=0.
```

```
Z=0.
```

```
CALL DATA
```

```
DO 100 II=15,60,5
```

```
I1=II
```

```
CALL EST(R,B,P1,I1)
CALL TEST(I1,P2,R,B,TX)
IF ( J1 .LT. 100 ) GO TO 60
IF ( J1 .EQ. 100 ) GO TO 70
H(I1,101)=TX
IF (H(I1,101) .LE. H(I1,100)) GO TO 100
IF (H(I1,101) .LE. H(I1,80)) GO TO 50
IF (H(I1,101) .LE. H(I1,60)) GO TO 40
IF (H(I1,101) .LE. H(I1,40)) GO TO 30
IF (H(I1,101) .LE. H(I1,20)) GO TO 20
M1=21
M2=100
CALL ORDER1(I1,M1,M2)
CALL ORDER2(I1,M1)
GO TO 100
20 M1=41
M2=100
CALL ORDER(I1,M1,M2)
CALL ORDER2(I1,M1)
GO TO 100
30 M1=61
M2=100
CALL ORDER1(I1,M1,M2)
CALL ORDER2(I1,M1)
GO TO 100
40 M1=81
```



```
M2=100

CALL ORDER1(I1,M1,M2)

CALL ORDER2(I1,M1)

GO TO 100

50 M1=101

CALL ORDER2(I1,M1)

GO TO 100

60 H(I1,J1)=TX

GO TO 100

70 H(I1,J1)=TX

DO 80 K3=1,99

DO 80 K1=K3,99

IF (H(I1,K3) .GE. H(I1,K1+1)) GO TO 80

CHANGE=H(I1,K3)

H(I1,K3)=H(I1,K1+1)

H(I1,K1+1)=CHANGE

80 CONTINUE

100 CONTINUE

DO 150 II=15,60,5

DO 150 L=50,100,25

POINT=H(I1,L)

WRITE(6,120)POINT,L,II,I,J

120 FORMAT(1H0,E16.8,' IS THE ',13,'TH LARGEST VALUE AT THE TRUNCA
TION VALUE ',12,'(',I1,',',I1,')')

150 CONTINUE

200 CONTINUE
```

STOP

END

SUBROUTINE DATA

COMMON X(100),N,H(60,101),LARGE,T,U,V,Y,Z

TEMP=0.0

DO 10 I=1,N

VV=RANDOM(LARGE)

UU=1.0-(1.0-TEMP)*VV**(1.0/(N-1+1.0))

X(I)=-ALOG(1-UU)

10 TEMP=UU

RETURN

END

SUBROUTINE EST(XR,XB,A,K)

COMMON X(100),N,H,(60,101),LARGE,T,U,V,Y,Z

XN=100.

IF (K .EQ. 15) GO TO 2

L=K-4

GO TO 3

2 L=1

3 M=K

DO 10 I=L,M

XI=I

10 T=T+EXP(A*(100.-XI)/100.)


```

DO 20 I=L,M

XI=I

Q=-ALOG(1-(XI-.5)/XN)

C=EXP(A*(100.-XI)/100.)

D=ALOG(Q)

E=ALOG(X(I))

U=U+C*D*E

V=V+C*E

Y=Y+C*D

20 Z=Z+C*E*E

XR=(U-V*Y/T)/(Z-V*V/T)

XB=EXP((V-Y/XR)/T)

RETURN

END

SUBROUTINE TEST(I,A,RR,BB,TST)

COMMON X(100),N,H,(60,101),LARGE,T,U,V,Y,Z

XN=100.

TST=0.

DO 10 M=1,I

XM=M

XI=I

F1=1-(XM-.5)/XN

F2=ESP(-(X(M)/BB)**RR)

F=F1-F2

```

```
W=EXP(A*XM/100.)  
10 TST=TST+W*F*F/XI  
RETURN  
END
```

```
SUBROUTINE ORDER1(I,J,K)  
COMMON X(100),N,H(60,101)  
DIMENSION TEMP(100)  
DO 10 L=J,K  
10 TEMP(L)=H(I,J)  
DO 20 L=J,K-1  
20 H(I,L+1)=TEMP(L)  
H(I,J)=H(I,101)  
RETURN  
END
```

```
SUBROUTINE ORDER2(I,J)  
COMMON X(100),N,H(60,101)  
CHANGE=H(I,J)  
H(I,J)=H(I,J-1)  
H(I,J-1)=CHANGE  
DO 10 L=1,19  
K=J-L  
IF (H(I,K) .LT. H(I,K-1)) GO TO 20  
CHANGE=H(I,K)
```


$H(I,K)=H(I,K-1)$

10 $H(I,K-1)=CHANGE$

20 RETURN

END

Appendix B

MAIN PROGRAM

```
COMMON X(100),N,LARGE,T,U,V,Y,Z

DIMENSION A(3),H(10),TIME(3),F(3),SUM(3,3,3),SS(3,3,3),BIAS(3,3,3)
1,VAR(3,3,3),SM(3,3,3)

N=100

LARGE=5114369

A(1)=3.

A(2)=4.5

A(3)=6.

TIME(1)=0.27

TIME(2)=1.

TIME(3)=3.

DO 10 MM=1,3

10 F(MM)=EXP(-(0.3)*TIME(MM))+EXP(-(0.8)*TIME(MM))-EXP(-TIME(MM))

DO 100 I=1,3

DO 100 J=1,3

P1=A(I)

P2=A(J)

DO 20 MM=1,3

SUM(I,J,MM)=0.

SM(I,J,MM)=0.

20 SS(I,J,MM)=0.

READ(5,15)(H(IJ),IJ=1,10)

15 FORMAT(2(5E15.8/))
```



```
DO 100 JJ=1,500

J1=JJ

T=0.

U=0.

V=0.

Y=0.

Z=0.

NN=0

CALL DATA

DO 30 II=15,60,5

II=II

NN=NN+1

CALL EST(R,B,P1,II)

CALL TEST(II,R,B,TX)

M1=(II-10)/5

IF ( TX .GE. H(M1) ) GO TO 40

PR=R

30 PB=B

40 NX=NN

IF ( NX .NE. 1 ) GO TO 50

TR=R

TB=B

GO TO 60

50 TR=R

TB=B

60 DO 70 MM=1,3
```

```

REL=EXP(-(TIME(MM)/TB)**TR)

XX=REL-F(MM)

SUM(I,J,MM)=SUM(I,J,MM)+XX

SM(I,J,MM)=SM(I,J,MM)+REL

70 SS(I,J,MM)=SS(I,J,MM)+REL*REL

IF ( J1 .EQ. 1 ) GO TO 100

XJ1=J1

DO 90 MM=1,3

BIAS(I,J,MM)=SUM(I,J,MM)/XJ1

VAR(I,J,MM)=(SS(I,J,MM)-SM(I,J,MM)*SM(I,J,MM)/XJ1)/(XJ1-1.)

80 FORMAT(1H0,'BIAS=',E16.8,';VARIANCE=',E16.8,';AT (' ,I2,',',I2,'
1,',I2,')',',',JJ=',I3,2X,E16.8,2X,E16.8)

90 CONTINUE

100 CONTINUE

STOP

END

SUBROUTINE DATA

COMMON X(100),N,LARGE,T,U,V,Y,Z

DIMENSION VV(2),UU(2),XX(2)

DO 10 I=1,N

DO 20 J=1,2

VV(J)=RANDOM(LARGE)

20 UU(J)=1.0-(1.0-TEMP)*VV(J)**(1.0/(N-1+1.))

```



```
XX(1)=- (5./4.)*ALOG(1-UU(1))  
XX(2)=-5.*ALOG(1-UU(2))  
J=1  
IF ( XX(2) .GE. XX(1) ) J=2  
X(I)=XX(J)  
10 TEMP=UU(J)  
  
RETURN  
  
END  
  
SUBROUTINE EST(XR,XB,A,K)  
COMMON X(100),N,LARGE,T,U,V,Y,Z  
( the same as in Appendix A )  
  
SUBROUTINE TEST(J,A,RR,BB,TST)  
COMMON X(100),N,LARGE,T,U,V,Y,Z  
( the same as in Appendix A )
```